FEDERAL PUBLIC SERVICE COMMISSION

COMPETITIVE EXAMINATION FOR RECRUITMENT TO POSTS IN BS-17 UNDER THE FEDERAL GOVERNMENT, 2013

Roll Number

PURE MATHEMATICS, PAPER-II

IME ALLOWED: THREE HOURS **MAXIMUM MARKS: 100**

OTE:(i) Candidate must write **Q. No.** in the **Answer Book** in accordance with **Q. No.** in the **Q. Paper.**
(ii) Attempt **FIVE** questions in all by selecting **THREE** questions from **SECTION-A** and **TWO** questions from **SECTION-B. ALL** questions carry **EQUAL** marks.
(iii) Extra attempt of any question or any part of the attempted question will not be considered.
(iv) **Use of calculator is allowed.**

SECTION-A

Q.1 (a) Let $\ell^{p}(p \geq 1)$ be the set of all sequences (ζ_j) of complex numbers such that the series $\sum_{j=1}^{n} |\zeta_j|^{p}$ converges. Let the real valued function $d: \ell^{p} \times \ell^{p} \to \mathrm{R}$ be defined by

$$d(x, y) = \left(\sum_{j=1}^{n} |\zeta_j - \eta_j|^{p}\right)^{1/p}$$

where $x = (\zeta_j)$ and $y = (\eta_j)$. Show that d is a metric on ℓ^{p}. (8)

(b) If d is the usual metric on R^n (the set of all ordered n-tuples of real numbers) then prove that (R^n, d) is a complete metric space. (7)

(c) Prove that the function $f: (\mathrm{X}, d_X) \to (\mathrm{Y}, d_y)$ is continuous $\Leftrightarrow f^{-1}(\mathrm{G})$ is closed in X whenever G is closed in Y. (5)

Q.2 (a) Prove that there exists no rational number x such that $x^2 = 2$. (5)

(b) Examine the continuity of f at $x = 0$ when

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ (5)

(c) Find the nth derivative of the function $e^x \ln x$. (5)

(d) Show that $f(x) = \dfrac{\ln(x+1)}{x}$ decreases on $]0, \infty[$. (5)

Q.3 (a) If $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$; find c of the Mean Value Theorem. (6)

(b) Examine the series $\sum_{n=1}^{\infty} \frac{n!}{n^2}$ for convergence or divergence. (5)

(c) Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ Converges or diverges. (5)

(d) Let $f(x) = |x|$. Check the differentiability of f at $x = 0$. (4)

If $u = \text{Sin}^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$. (5

(b) Find the percentage error in calculating the area of a rectangle when there is error of percent in measuring its sides. (5

(c) An open rectangular box is to be made from a sheet of cardboard 8dm by 5dm, by cutting equal squares from each corner and turning up the sides. Find the edge of the square which makes the volume maximum. (5)

(d) Find the asymptotes of the curve, $y = \frac{x^3+x-2}{x-x^2}$. (5)

Q.5 (a) Evaluate the double integral of $F(x, y) = x^2 + xy$, over the triangle with vertices (0, 0), (0, 1) and (1, 1). (5)

(b) Let f be Riemann integrable on $[a, b]$. Prove that $|f|$ is also Riemann integrable on $[a, b]$ and

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \qquad (8)$$

(c) Examine the convergence of the improper integral $\int_0^2 \frac{dx}{2x-x^2}$. (7)

SECTION-B

Q.6 (a) Solve the equation, $z^2 + (2i-3)z + 5 - i = 0$ (8)

(b) Prove that
$\text{Cos}^{-1}(\cos\theta + i\sin\theta) = \text{Sin}^{-1}(\sqrt{\sin\theta}) + i\,\ell n(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$ (8)

(c) If $w = f(z)$ is differentiable then prove that $f(z)$ is continuous. (4)

Q.7 (a) Prove that the essential characteristics ~~for~~ a function $f(z)$ to be analytic is that $\frac{\partial f}{\partial \bar{z}} = 0$. (5)

(b) If $u(x, y)$ is a harmonic function then prove that it satisfies the differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$. (5)

(c) Show that the function $f(z) = \cos(z + \frac{1}{z})$ can be expanded as a Laurent's series,

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right),$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\cos\theta)\cos n\theta \, d\theta$ (10)

Q.8 (a) Prove that $\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = \frac{2\pi}{e^a},\ a > 0$ (8)

(b) Prove that $\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ (6)

(c) Let $f(z)$ be analytic on a closed contour $C : |z - a| = r$. If $|f(z)| \le M$ then prove that $|f^n(a)| \le \frac{n!}{r^n}M$. (6)

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